Chapter 3

We have already encountered sequences in an arbitrary metric space setting. In this, chapter we are primerily concerned with real-valued sequences. Before reading further, the reader is strongly encouraged to review the discussion on Cauchy sequences, convergent sequences, bounded sequences, and subsequences.

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of real (or complex) numbers. Recall that the set of all points p_n $(n=1,2,3,\dots)$ is the range of $\{p_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{p_n\}$ is said to be bounded if its range is bounded.

Ex, Consider the following sequences of complex numbers:

(a) If $s_n = 1/n$, then $\lim_{n\to\infty} s_n = 0$; the range is infinite, and the sequence is bounded.

(b) If $s_n = n^2$, the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.

(c) If $s_n = 1 + (-1)^n/n$, the sequence $\{s_n\}$ converges to 1, is bounded, and has infinite range.

(d) If $s_n = i^n$, the sequence $\{s_n\}$ is divergent, is bounded, and has finite range.

(e) If $s_n = 1$ $(n=1,2,3,\ldots)$ then $\{s_n\}$ converges to 1, is bounded, and has finite range.

For sequences in $\mathbb{R}^m$ we can study the relation between convergence, on the one hand, and algebraic operations on the other. We first consider sequences of complex numbers.

Thm: Suppose $\{s_n\}$, $\{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$;

(b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty} (c + s_n) = c + s$, for any number c;

(c) $\lim_{n\to\infty} s_n t_n = st$;

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$ $(n=1,2,3,\ldots)$ and $s \neq 0$.

Proof:

(a) Given $\epsilon > 0$, there exist integers N_1, N_2 such that

$$n \geq N_1 \text{ implies } |s_n - s| < \frac{\epsilon}{2}$$

$$n \geq N_2 \text{ implies } |t_n - t| < \frac{\epsilon}{2}$$

If $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$|(s_n+t_n)-(s+t)| \leq |s_n-s|+|t_n-t| < \epsilon$$

This proves (a). The proof of (b) is trivial.

(c) We use the identity

$$s_n t_n - st = (s_n-s)(t_n-t)+s(t_n-t)+t(s_n-s). \qquad (1)$$

Given $\epsilon>0$, there are integers N_1, N_2 such that

$$n \geq N_1 \text{ implies } |s_n-s| < \sqrt{\epsilon}$$

$$n \geq N_2 \text{ implies } |t_n-t| < \sqrt{\epsilon}$$

If we take $N=\max\{N_1, N_2\}$, $n\geq N$ implies

$$|(s_n-s)(t_n-t)| < \epsilon$$

so that

$$\lim_{n\to\infty} (s_n-s)(t_n-t) = 0$$

We now apply (a) and (b) to (1), and conclude that

$$\lim_{n\to\infty} (s_n t_n - st) = 0$$

(d) Choosing m such that $|s_n-s| < \frac{1}{2}|s|$ if $n \geq m$, we see that

$$||s_n|-|s|| \leq |s_n-s| < \frac{1}{2}|s|$$

or

$$-\frac{1}{2}|s| \leq |s_n|-|s| \leq \frac{1}{2}|s| \quad \text{implying that}$$

$$\tfrac{1}{2}|s| < |s_n| < \tfrac{3}{2}|s|$$

In particular

$$|s_n| > \tfrac{1}{2}|s| \quad (n \geq m)$$

Given $\epsilon > 0$, there is an integer $N > m$ such that $n \geq N$ implies

$$|s_n - s| < \tfrac{1}{2}|s|^2 \epsilon$$

Hence, for $n \geq N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s| < \epsilon.$$

Thm: (a) Suppose $x_n \in \mathbb{R}^k$ $(n = 1, 2, 3, \ldots)$ and

$$x_n = (\alpha_{1,n}, \ldots, \alpha_{k,n})$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to $x = (\alpha_1, \ldots, \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k)$$

(b) Suppose $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty} (x_n + y_n) = x + y, \quad \lim_{n\to\infty} x_n \cdot y_n = x \cdot y, \quad \lim_{n\to\infty} \beta_n x_n = \beta x.$$

Proof: (a) If $x_n \to x$, the inequalities

$$|\alpha_{j,n} - \alpha_j| \leq \|x_n - x\|_2 \qquad (2)$$

which follow immediately from the definition of the norm in $\mathbb{R}^k$, show that (2) holds.

Conversely, if (2) holds, then for $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|\alpha_{j,n} - \alpha_j| < \frac{\epsilon}{k} \quad (1 \leq j \leq k)$$

Hence $n \geq N$ and the triangle inequality imply

$$\|x_n - x\|_2 \leq \sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j| < \sum_{j=1}^{k} \frac{\epsilon}{k} = \epsilon$$

so that $x_n \to x$. This proves (a)

Part (b) follows from (a) and the previous theorem.

Upper and Lower Limits

So far we know that monotone, bounded sequences converge, and that any convergent sequence is necessarily bounded. (Why?) These two facts together raise the question: Does every bounded sequence converge? Of course not. But just how "far" from convergent is a typical bounded sequence? To answer this, we will want to broaden our definition of limit. First a few easy observations.

Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers, and consider the sequences:

$$t_n = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\} \text{ and } T_n = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

Then $\{t_n\}$ increases, $\{T_n\}$ decreases, and $\inf_{k \in \mathbb{N}} a_k \leq t_n \leq T_n \leq \sup_{k \in \mathbb{N}} a_k$

for all n. (Why?) Thus we may speak of $\lim_{n\to\infty} t_n$ as the "lower limit" and $\lim_{n\to\infty} T_n$ as the "upper limit" of our original sequence $\{a_n\}$.

Now these same considerations are meaningful even if we start with an unbounded sequence $\{a_n\}$, although in that case we will have to allow the values $\pm\infty$ for at least some of the t_n's or T_n's (possibly both). That is, if we permit comparisons to $\pm\infty$, then the t_n's still increase and the T_n's still decrease. Of course we will want to use $\sup_{n\in\mathbb{N}} t_n$ and $\inf_{n\in\mathbb{N}} T_n$ in place of $\lim_{n\to\infty} t_n$ and $\lim_{n\to\infty} T_n$, since "sup" and "inf" have more or less obvious extensions to subsets of the extended real number system $[-\infty,\infty]$ whereas "lim" does not. Even so, we are sure to get caught saying something like "$\{t_n\}$ converges to $+\infty$". But we will pay a stiff penalty for too much rigor here; even a simple fact could have a tediously long description. For the remainder of this chapter you are encouraged to interpret words such as "limit" and "converges" in this looser sence.

Given any sequence of real numbers $\{a_n\}$, we define

$$\liminf_{n\to\infty} a_n = \varliminf_{n\to\infty} a_n = \sup_{n\geq 1}\{\inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}\}$$

and

$$\limsup_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = \inf_{n\geq 1}\{\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}\}$$

That is, $\liminf_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} t_n$ $(= \lim_{n\to\infty} t_n$ if $\{a_n\}$ is bounded from below) and $\limsup_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} T_n$ $(= \lim_{n\to\infty} T_n$ if $\{a_n\}$ is bounded from above) The name "lim inf" is short for "limit inferior" while "lim sup" is short for "limit superior".

Ex. (a) Let $a_n = \frac{1}{n}$. Then $t_n = \inf_{k \geq n} a_k = 0$ and $T_n = \sup_{k \geq n} a_k = \frac{1}{n}$. Clearly $\lim_{n\to\infty} t_n = \lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} T_n = 0$. Thus $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = 0$.

(b) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $\{1, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{4}, \ldots\}$. That is $a_{2k} = \frac{k-1}{k}$ and $a_{2k-1} = \frac{1}{k}$. Then $t_n = 0$ and $T_n = 1$ (Why?) Clearly $\liminf_{n\to\infty} a_n = 0 < 1 = \limsup_{n\to\infty} a_n$.

(c) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $\{1, -1, 2, -2, 3, -3, \ldots\}$. Then $\liminf_{n\to\infty} a_n = -\infty < \infty = \limsup_{n\to\infty} a_n$ (Why?)

(d) Let $a_n = \dfrac{(-1)^n}{1 + \frac{1}{n}}$.

Then $\liminf_{n\to\infty} a_n = -1$, while $\limsup_{n\to\infty} a_n = 1$.

Some Special Sequences

We shall now compute the limits of some sequences which occur frequently. But before we do this let's review the binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$. Recall also that $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n > 0$

and $0! = 1$

Thus $(x+y)^2 = y^2\binom{2}{0} + xy\binom{2}{1} + x^2\binom{2}{2} = y^2\frac{2!}{0!\,2!} + xy\frac{2!}{2!\,1!} +$
$+ x^2\frac{2!}{2!\,0!} = y^2 + 2xy + x^2$

$$(x+y)^3 = y^3\binom{3}{0} + xy^2\binom{3}{1} + x^2y\binom{3}{2} + x^3\binom{3}{3} =$$
$$= y^3\frac{3!}{0!\,3!} + xy^2\frac{3!}{1!\,2!} + x^2y\frac{3!}{2!\,1!} + x^3\frac{3!}{3!\,0!} =$$
$$= y^3 + 3xy^2 + 3x^2y + x^3.$$

To proceed we will also need the following: If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$

Thm: (a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(d) If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

Proof:

(a) Take $n > (1/\varepsilon)^{1/p}$. (Note that the archimedean property of the real number system is used here.)

(b) If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and, by the binomial theorem,

$$1 + nx_n \leq (1+x_n)^n = p$$

so that

$$0 < x_n \leqslant \frac{p-1}{n}$$

Hence $x_n \to 0$. If $p = 1$, (b) is trivial, and if $0 < p < 1$, the result is obtained by taking reciprocals.

(c) Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \geqslant 0$, and, by the binomial theorem,

$$n = (1 + x_n)^n \geqslant \frac{n(n-1)}{2} x_n^2$$

Hence

$$0 \leqslant x_n \leqslant \sqrt{\frac{2}{n-1}} \quad (n \geqslant 2).$$

(d) Let $k > \alpha$, $k > 0$. For $n > 2k$, observe that

$$n - k + 1 - \frac{n}{2} = \frac{n}{2} - k + 1 > k - k + 1 = 1 > 0. \text{ Thus } n - k + 1 > \frac{n}{2}.$$

Notice that

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\ldots(n-k+1)}{k!} p^k > \frac{n^k}{2^k k!} p^k \quad (\text{why?})$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \quad (n > 2k)$$

Since $\alpha - k < 0$, $n^{\alpha - k} \to 0$ by (a).

(e) Take $\alpha = 0$ in (d).

Series

In the remainder of this chapter, all sequences and series under consideration will be complex-valued, unless contrary is explicitly stated.

Def: Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^{q} a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \dots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k$$

For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \dots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n$$

We call this expression an infinite series, or just a series. The numbers s_n are called the partial sums of the series. If $\{s_n\}$ converges to s, we say that the series converges, and write

$$\sum_{n=0}^{\infty} a_n = s.$$

The number s is called the sum of the series; but it should be clearly understood that s is the limit of a sequence of sums, and is not obtained simply by addition.

If $\{s_n\}$ diverges, the series is said to diverge.

Sometimes, for convenience of notation, we shall consider series of the form

$$\sum_{n=0}^{\infty} a_n .$$

And frequently, when there is no possible ambiguity, or when the distinction is immaterial, we shall simply write $\sum a_n$.

It is clear that every theorem about sequences can be stated in terms of series (putting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n>1$), and vice versa. But it is nevertheless useful to consider both concepts.

The Cauchy criterion (i.e. every convergent sequence is Cauchy) can be restated in the following form:

Thm: $\sum a_n$ converges if and only if for every $\epsilon>0$ there is an integer N such that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \epsilon$$

if $m \geq n \geq N$.

In particular, by taking $m=n$, the above expression becomes

$$|a_n| \leq \epsilon .$$

Also notice that if $n=N$ and $m \to \infty$, the expression becomes

$$\left| \sum_{n=N}^{\infty} a_n \right| \leq \epsilon$$

Proof: In $\mathbb{R}$ and in $\mathbb{C}$ every Cauchy sequence converges. (why?) Thus the sequence S_n of partial sums is convergent if and only if it is Cauchy. Now, S_n is Cauchy if for any $\epsilon > 0$ there is some N such that $n, m \geq N$ implies $|S_n - S_m| < \epsilon$. If $m \geq n$
$|S_n - S_m| = |S_m - S_n| = \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{m} a_k \right|$.

Corollary: If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

The condition $a_n \to 0$ is not, however, sufficient to ensure convergence of $\sum a_n$. For instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges as we'll demonstrate later.

Corollary: A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

We now turn to a convergence test of a different nature, the so-called "comparison test".

Thm: (a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Note that (b) applies only to series of nonnegative terms a_n

Proof: Given $\epsilon > 0$, there exists $N \geq N_0$ such that $m \geq n \geq N$ implies

$$\sum_{k=n}^{m} c_k \leq \epsilon$$

by the Cauchy criterion. Hence

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k \leq \epsilon$$

and (a) follows.

Next, (b) follows from (a), for if $\sum a_n$ converges, so must $\sum d_n$.

The comparison test is a very useful one; to use it efficiently, we have to become familiar with a number of series of nonnegative terms whose convergence or divergence is known.

Series of Nonnegative Terms

Thm: If $0 \leq x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If $x \geq 1$, the series diverges.

Proof: If $x \neq 1$,

$$s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

The result follows if we let $n \to \infty$. For $x = 1$, we get

$$1 + 1 + 1 + \ldots$$

which evidently diverges.

In many cases which occur in applications, the terms of the series decrease monotonically. The following theorem of Cauchy is therefore of particular interest. The striking feature of the theorem is that a rather "thin" subsequence of $\{a_n\}$ determines the convergence or divergence of $\sum a_n$.

Thm: Suppose $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. Then the the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$$

converges.

Proof: Since the series under consideration has nonnegative terms, it suffices to consider boundedness of the partial sums.

Let
$$s_n = a_1 + a_2 + \ldots + a_n$$
$$t_k = a_1 + 2a_2 + \ldots + 2^k a_{2^k}$$

For $n < 2^k$,

$$s_n \leq a_1 + (a_2 + a_3) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1})$$
$$\leq a_1 + 2a_2 + \dots + 2^k a_{2^k} = t_k$$

so that $\quad s_n \leq t_k$.

On the other hand, if $n > 2^k$

$$S_n \geq a_1 + a_2 + (a_3 + a_4) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k})$$
$$\geq \tfrac{1}{2}a_1 + a_2 + 2a_4 + \dots + 2^{k-1}a_{2^k} = \tfrac{1}{2}t_k$$

so that $\; 2s_n \geq t_k$

Thus the sequences $\{s_n\}$ and $\{t_k\}$ are either both bounded or both unbounded. This completes the proof.

Corollary: $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof: If $p \leq 0$, $\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p} = \infty$ and the series diverges

If $p > 0$, the sequence $\frac{1}{n^p}$ decreases and the above theorem applies and we are led to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}$$

Now, $2^{1-p} < 1$ if and only if $1-p < 0$, and the result follows by comparison with the geometric series $\sum x^k$, where $x = 2^{1-p}$.

As further application, we prove:

Corollary: If $p > 1$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges; if $p \le 1$, the series diverges

Proof: The monotonicity of the logarithmic function implies that $\{\ln n\}$ increases. Hence $\{\frac{1}{n \ln n}\}$ decreases, and we can apply the above theorem. This leads us to the series

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\ln 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \ln 2)^p} = \frac{1}{(\ln 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

and the conclusion follows.

This procedure may evidently be continued. For instance,

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$$

diverges, whereas

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^2}$$

converges.

The Number e

Def: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Since $s_n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \dots + \frac{1}{1\cdot 2 \cdots n}$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} < 3$$

the series converges, and the definition makes sense. In fact, the series converges very rapidly and allows us to compute e with great accuracy.

It is of interest to note that e can also be defined by means of another limit process; the proof provides a good illustration of operations with limits.

Thm: $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

Proof: Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad t_n = \left(1+\frac{1}{n}\right)^n.$$

Clearly the sequence s_n is monotonically increasing. To see that t_n is monotonically increasing as well, observe that by the binomial theorem $t_n = \left(1+\frac{1}{n}\right)^n > \left(1+n\frac{1}{n}\right) = 2$. In fact, if $a > -1$, $a \neq 0$, then $(1+a)^n > 1+na$.

Now $$\frac{t_{n+1}}{t_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \left(1+\frac{1}{n}\right)\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\cdot\left(\frac{n^2+2n}{(n+1)^2}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1}$$

$$> \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right) = 1$$

Thus $t_{n+1} > t_n$ and t_n is increasing as desired.

By the binomial theorem

$$t_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$
$$\leq S_n < e$$

Thus $\{t_n\}_{n=1}^{\infty}$ is also a bounded sequence. Hence $\lim_{n\to\infty} t_n = \alpha \leq e$.

Next, if $n \geq m$,

$$t_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\dots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{m-1}{n}\right)+\dots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$
$$\geq 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\dots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{m-1}{n}\right)$$

Thus

$$\alpha > t_n \geq 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\dots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{m-1}{n}\right)$$

Let $n\to\infty$, keeping m fixed. We get

$$\alpha > 1+1+\frac{1}{2!}+\dots+\frac{1}{m!} = S_m$$

This means that

$$\alpha \geq \lim_{m\to\infty} S_m = e$$

Hence $\alpha = e$.

The rapidity with which the series $\sum \frac{1}{n!}$ converges can be estimated as follows: If S_n has the same meaning as above, we have

$$e - S_n = \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\dots < \frac{1}{(n+1)!}\left(1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\dots\right) = \frac{1}{n!\,n}$$

so that

$$0 < e - S_n < \frac{1}{n!\,n}$$

Thus s_{10}, for instance, approximates e with an error less than 10^{-7}.
The inequality is of theoretical interests as well, since it enables us to prove the irrationality of e very easily.

Thm: e is irrational

Proof: Suppose e is rational. Then $e = p/q$, where p and q are positive integers. By the above inequality

$$0 < q!(e - s_q) < \frac{1}{q}$$

By our assumption, $q!e$ is an integer. Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!}\right)$$

is an integer, we see that $q!(e - s_q)$ is an integer.
But then

$$q!(e - s_q)$$

is also an integer, which implies the existence of an integer between 0 and 1. We have thus reached a contradiction.

The Root and Ratio Tests

Thm (Root Test): Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

Then
(a) if $\alpha < 1$, $\sum a_n$ converges
(b) if $\alpha > 1$, $\sum a_n$ diverges
(c) if $\alpha = 1$, the test gives no information.

Proof: If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq N$. That is, $n \geq N$ implies

$$|a_n| < \beta^n$$

Since $0 < \beta < 1$, $\sum \beta^n$ converges. Convergence of $\sum a_n$ follows now from the comparison test.

If $\alpha > 1$, then, again, there is a sequence $\{n_k\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \to \alpha$$

Hence $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \mapsto 0$, necessary for convergence of $\sum a_n$, does not hold.

To prove (c), we consider the series

$$\sum \frac{1}{n}, \sum \frac{1}{n^2}$$

Since $\sqrt[n]{\frac{1}{n}} \mapsto 1$ and $\sqrt[n]{\frac{1}{n^2}} \mapsto 1$ and since one series diverges while the other converges, we see that $\alpha = 1$ gives no information.

Thm (Ratio Test): The series $\sum a_n$

(a) converges if $\displaystyle\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Proof: If condition (a) holds, we can find $\beta < 1$, and an integer N, such that

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta$$

for $n \geq N$. In particular,

$$|a_{N+1}| < \beta |a_N|$$

$$|a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|$$

$$\vdots$$

$$|a_{N+p}| < \beta^p |a_N|$$

That is,

$$|a_n| < |a_N| \beta^{-N} \beta^n$$

for $n \geq N$, and (a) follows from the comparison test, since $\sum \beta^n$ converges.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, it is easily seen that the condition $a_n \to 0$ does not hold, and (b) follows.

Note: The knowledge that $\lim a_{n+1}/a_n = 1$ implies nothing about the convergence of $\sum a_n$. The series $\sum 1/n$ and $\sum 1/n^2$ demonstrate this.

Ex, (a) Consider the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\dots$$

for which

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0$$

$$\liminf_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[2n]{\frac{1}{3^n}}=\frac{1}{\sqrt{3}}$$

$$\limsup_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[2n]{\frac{1}{2^n}}=\frac{1}{\sqrt{2}}$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2}\left(\frac{3}{2}\right)^n=\infty$$

The root test indicates convergence; the ratio test does not apply.

(b) The same is true for the series

$$\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{1}{64}+\dots$$

where

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{8}$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=2$$

but

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{1}{2}.$$

Remarks: The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than n^{th} roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of the theorem below.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \to \infty$.

Thm: For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n}$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

Proof: We shall prove the second inequality; the proof of the first is quite similar. Put

$$\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that

$$\frac{c_{n+1}}{c_n} \leq \beta$$

for $n \geq N$. In particular, for any $p > 0$,

$$c_{N+k+1} \leq \beta c_{N+k} \qquad (k = 0, 1, \ldots, p-1)$$

Multiplying these inequalities, we obtain

$$c_{N+p} \leq \beta^p c_N,$$

or
$$c_n \leq c_N \beta^{-N} \beta^n \quad (n \geq N)$$

Hence
$$\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N}}\, \beta.$$

so that
$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \beta.$$

because $\lim\limits_{n\to\infty} \sqrt[n]{c_N \beta^{-N}} = 1.$

Since $\limsup\limits_{n\to\infty} \sqrt[n]{c_n} \leq \beta$ for every $\beta > \alpha$, we have

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \alpha.$$

Power Series

Def: Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a power series. The numbers c_n are called the coefficients of the series; z is a complex number.
In general, the series will converge or diverge, depending on the choice of z. More specifically, with every power series there is associated a circle, the circle of convergence, such that the

power series converges if z is in the interior of the circle and diverges if z is in the exterior (to cover all cases, we have to consider the plane as the interior of a circle of infinite radius, and a point as a circle of radius zero). The behavior on the circle of convergence is much more varied and cannot be described so simply.

Thm: Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}$$

(If $\alpha=0$, $R=+\infty$, if $\alpha=+\infty$, $R=0$). Then $\sum c_n z^n$ converges if $|z|<R$, and diverges if $|z|>R$

Proof: Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\sum c_n z^n$.

Ex,

(a) The series $\sum n^n z^n$ has $R=0$

(b) The series $\sum \frac{z^n}{n!}$ has $R=\infty$ (In this case the ratio test is easier to apply than the root test.)

(c) The series $\sum z^n$ has $R=1$. If $|z|=1$, the series diverges, since $\{z^n\}$ does not tend to 0 as $n\to\infty$.

(d) The series $\sum \frac{z^n}{n}$ has $R=1$. It diverges at $z=1$. It converges for all other z with $|z|=1$ (The last assertion will be proved later).

(e) The series $\sum \frac{z^n}{n^2}$ has $R=1$. It converges for all z with $|z|=1$, by the comparison test, since $\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}$.

Summation by Parts

Thm: Given two sequences $\{a_n\}$, $\{b_n\}$, put

$$A_n = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1}=0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Proof:

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} = A_q b_q + \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} - A_{p-1} b_p.$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

The so-called "partial summation formula" is useful in the investigation of series of the form $\sum a_n b_n$, particularly when $\{b_n\}$ is monotonic.

Thm: Suppose

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots \geq 0$;

(c) $\lim_{n\to\infty} b_n = 0$

Then $\sum a_n b_n$ converges.

Proof: Choose M such that $|A_n| \leq M$ for all n. Given $\epsilon > 0$, there is an integer N such that $b_N \leq \epsilon/2M$. For $N \leq p \leq q$, we have

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \leq$$

$$\leq \sum_{n=p}^{q-1} |A_n||b_n - b_{n+1}| + |A_q||b_q| + |A_{p-1}||b_p|$$

$$\leq M\left(\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right)$$

$$= M\left(b_p - b_q + b_q + b_p\right) = 2Mb_p \leq 2Mb_N \leq \epsilon$$

Convergence now follows from the Cauchy criterion.

Thm: Suppose

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \ldots$

(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $(m = 1, 2, 3, \ldots)$

(c) $\lim_{n\to\infty} c_n = 0$

Then $\sum c_n$ converges.

Series for which (b) holds are called "alternating series".

Proof: $\sum c_n = \sum a_n b_n$ where $a_n = (-1)^{n+1}$, $b_n = |c_n|$. Notice that $A_n = \sum_{k=1}^{n} a_n$ is bounded with $|A_n| \leq 1$, and $b_n \geq b_{n+1}$. Thus the result follows by the previous theorem.

Thm: Suppose the radius of convergence of $\sum c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \ldots$, $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

Proof: Put $a_n = z^n$, $b_n = c_n$. Then

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1-z^{n+1}}{1-z}\right| \leq \frac{|1|+|z|^{n+1}}{|1-z|} =$$

$$= \frac{1+1}{|1-z|} = \frac{2}{|1-z|}$$

if $|z| = 1$, $z \neq 1$.

Absolute Convergence

The series $\sum a_n$ is said to converge absolutely if the series $\sum |a_n|$ converges.

Thm: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof: The assertion follows from the inequality

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k|$$

Plus the Cauchy criterion.

Remarks: For series of positive terms, absolute convergence is the same as convergence.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges non-absolutely. For instance, the series

$$\sum \frac{(-1)^n}{n}$$

converges nonabsolutely.

The comparison test, as well as the root and ratio tests, is really a test for absolute convergence, and therefore cannot give any information about non-absolutely convergent series. Summation by parts can sometimes be used to handle the latter. In particular, power series converge absolutely in the interior of the circle of convergence.

We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them term by term and we may change the order in which the additions are carried out, without affecting the sum of the series. But for nonabsolutely convergent series this is no longer true, and more care has to be taken when dealing with them.

Addition and Multiplication of Series

Thm: If $\sum a_n = A$, and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$, and $\sum c a_n = cA$, for any fixed c.

Proof: Let

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k$$

Then $A_n + B_n = \sum_{k=0}^{n} (a_k + b_k)$

Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, we see that

$$\lim_{n\to\infty} (A_n + B_n) = A + B$$

The proof of the second assertion is similar.

Thus two convergent series may be added term by term, and the resulting series converges to the sum of the two series. The situation becomes more complicated when we consider multiplication of two series. To begin with, we have to define the product. This can be done in several ways; we shall consider the so-called "Cauchy product."

Def: Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \ldots)$$

and call $\sum c_n$ the product of the two given series.

This definition may be motivated as follows. If we take two power series $\sum a_n z^n$ and $\sum b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n = (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots$$

$$= c_0 + c_1 z + c_2 z^2 + \dots$$

Setting $z=1$, we arrive at the above definition.

Ex. If

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k,$$

and $A_n \mapsto A$, $B_n \mapsto B$, then it is not at all clear that $\{C_n\}_{n=1}^{\infty}$ will converge to AB, since we do not have $C_n = A_n B_n$. The dependence of $\{C_n\}$ on $\{A_n\}$ and $\{B_n\}$ is quite a complicated one. We shall now show that the product of two convergent series may actually diverge

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$

converges by the alternating test. We form the product of this series with itself and obtain

$$\sum_{n=0}^{\infty} c_n = 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) - $$
$$ - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \dots$$

where $c_n = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}$

$$= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}$$

To estimate $|c_n|$, observe that the function

$$f(x) = (x+1)(n-x+1)\ , \ x \in [0,n]$$

is differentiable over $(0,n)$ with derivative

$$f'(x) = (n-x+1) - (x+1) = n - 2x = 0$$

when $x = \frac{n}{2}$.

Notice that $f''(x) = -2 < 0$ so $f''(\frac{n}{2}) < 0$, implying that $f(\frac{n}{2})$ is a local maximum.

Notice that $f(\frac{n}{2}) = (\frac{n}{2}+1)(n-\frac{n}{2}+1) = (\frac{n}{2}+1)^2 > \frac{n}{2}+1$

whereas $f(n) = f(0) = n+1 < \frac{n^2}{4} + n + 1 = (\frac{n}{2}+1)^2 = f(\frac{n}{2})$.

Thus $f(\frac{n}{2})$ is the absolute maximum. In particular, for $0 \le k \le n$

$(k+1)(n-k+1) \le (\frac{n}{2}+1)^2$ or $\sqrt{(k+1)(n-k+1)} \le \sqrt{(\frac{n}{2}+1)^2} = \frac{n}{2}+1$

Therefore

$$\frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \frac{1}{\frac{n}{2}+1} = \frac{2}{n+2}$$

Thus

$$|c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$$

which implies that

$$\limsup_{n\to\infty} |c_n| \geq \limsup_{n\to\infty} \frac{2(n+1)}{n+2} = 2$$

suggesting that $\sum c_n$ diverges by the divergence test.

In view of the next theorem, due to Mertens, we note that we have here considered the product of two nonabsolutely convergent series.

Thm: Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely

(b) $\sum_{n=0}^{\infty} a_n = A$

(c) $\sum_{n=0}^{\infty} b_n = B$

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \ldots)$

Then

$$\sum_{n=0}^{\infty} c_n = AB$$

That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Proof: Put

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k, \quad \beta_n = B_n - B.$$

Then

$$C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0)$$
$$= a_0 B_n + a_1 B_{n-1} + \ldots + a_n B_0$$
$$= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \ldots + a_n (B + \beta_0)$$
$$= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \ldots + a_n \beta_0$$

Put

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \ldots + a_n \beta_0$$

We wish to show that $C_n \mapsto AB$. Since $A_n B \mapsto AB$, it suffices to show that

$$\lim_{n \to \infty} \gamma_n = 0$$

Put $\alpha = \sum_{n=0}^{\infty} |a_n|$

[It is here that we use (a)]. Let $\epsilon > 0$ be given. By (c), $\beta_n \mapsto 0$. Hence we can choose N such that $|\beta_n| \leq \epsilon$ for $n \geq N$, in which case

$$|\gamma_n| \leq |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N+1} + \ldots + \beta_n a_0|$$
$$\leq |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + |\beta_{N+1}| \sum_{k=n-N+1}^{n-N+1} |a_k| + \ldots + |\beta_n| \sum_{k=0}^{0} |a_k|$$
$$\leq |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + \epsilon \alpha$$

Keeping N fixed, and letting $n \to \infty$, we get

$$\limsup_{n \to \infty} |\gamma_n| \leq \limsup_{n \to \infty} |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + \epsilon \alpha = \epsilon \alpha$$

Since ϵ is arbitrary, $C_n \to AB$ (because $\gamma_n \mapsto 0$) as desired